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# ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CERTAIN NONLINEAR PROBLEMS OF HYDRODYNAMICS

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Periodic solutions of heat conduction equations with boundary conditions of the relay kind are found for a finite interval, and the behavior of such solutions at unlimited time increase is analyzed. Periodic solutions of heat conduction equations with nonlinear boundary conditions were considered in [1 - 4, 10], while in [5, 6] periodic solutions of nonhomogeneous heat conduction equations with their right-hand sides nonlinear with respect to the unknown functions are presented, and the asymptotic behavior of related initial problems is analyzed. Solutions of this kind define self-oscillating processes occurring in various branches of hydrodynamics (theory of filtration and diffusion [3 - 6]).

**1.** The problem reduces to finding the periodic solution of equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

in the finite region $-l < x < 0$ with boundary conditions

$$\frac{\partial u(-l, t)}{\partial x} = \begin{cases} h_1 u(-l, t) + q_1 & \text{for} \quad u(-l, t) < u_* \\ h_2 u(-l, t) + q_2 & \text{for} \quad u(-l, t) > u_{**} \end{cases} \tag{1.2}$$

$$(u_* > u_{**},\ h_1 > 0,\ h_2 > 0,\ q_2 > q_1)$$

$$u(0, t) = 0$$

We set $u(-l, t) = u_*$ at $t = T_1$ and $u(-l, t) = u_{**}$ at $t = T$, with $u = u_1(x, t)$ for $0 \leqslant t \leqslant T_1$ and $u = u_2(x, t)$ for $T_1 \leqslant t \leqslant T$, and seek the solution of this problem in the form of series

$$u_1(x,t)=\frac{q_1x}{1+h_1l}+\sum_{k=1}^{\infty}C_k\exp(-\lambda_{k_1}{}^2t)\sin\alpha_{k_1}x \tag{1.3}$$

$$u_2(x,t)=\frac{q_2x}{1+h_2l}+\sum_{k=1}^{\infty}D_k\exp[-\lambda_{k_2}{}^2(t-T_1)]\sin\alpha_{k_2}x$$

where $\lambda_{k_i}$ $(i=1,\ 2)$ are the roots of equations

$$\operatorname{tg}\frac{\lambda_{k_i}l}{a}=-\frac{1}{h_i}\frac{\lambda_{k_i}}{a}\qquad\left(\alpha_{k_i}=\frac{\lambda_{k_i}}{a}\right) \tag{1.4}$$

The inequalities

$$^1/_2\pi(2k-1)<\alpha_{k_i}l<\pi k\qquad(k=1,2,\ldots)$$

are valid, and $\alpha_{k_i}l\to{}^1/_2\pi(2k-1)$ when $k\to\infty$.

The coefficients $C_k$ and $D_k$ $(k=1,\ 2,\ \ldots)$ are determined by the conditions of continuity of solution $u_1(x,\ T_1)=u_2(x,\ T_1)$, $u_2(x,\ T)=u_1(x,\ 0)$. From (1.3) we obtain the equalities

$$\frac{q_1}{1+h_1l}x+\sum_{k=1}^{\infty}C_k\exp(-\lambda_{k_1}{}^2T_1)\sin\alpha_{k_1}x=\frac{q_2}{1+h_2l}+\sum_{k=1}^{\infty}D_k\sin\alpha_{k_2}x \tag{1.5}$$

$$\frac{q_2}{1+h_2l}x+\sum_{k=1}^{\infty}D_k\exp[-\lambda_{k_2}{}^2(T-T_1)]\sin\alpha_{k_2}x=$$

$$=\frac{q_1}{1+h_1l}x+\sum_{k=1}^{\infty}C_k\sin\alpha_{k_1}x$$

Constants $T_1$ and $T$ are the smallest roots of equations $u(-l,\ T_1)=u_*$, $u(-l,\ T)=u_{**}$. By virtue of (1.3) these equations become

$$-\frac{q_1l}{1+h_1l}-\sum_{k=1}^{\infty}C_k\exp(-\lambda_{k_1}{}^2T_1)\sin\alpha_{k_1}l=u_* \tag{1.6}$$

$$-\frac{q_2l}{1+h_2l}-\sum_{k=1}^{\infty}D_k\exp[-\lambda_{k_2}{}^2(T-T_1)]\sin\alpha_{k_2}l=u_{**}$$

We in troduce notation

$$B=\frac{q_1}{1+h_1l}-\frac{q_2}{1+h_2l},\qquad\beta_{k_1}=\exp(-\lambda_{k_1}{}^2T_1) \tag{1.7}$$

$$\gamma_{k_2}=\exp[\lambda_{k_2}{}^2(T-T_1)],\qquad A_k=\beta_{k_1}C_k,\qquad B_k=\gamma_{k_2}D_k$$

Equations (1.5) reduce then to

$$\sum_{k=1}^{\infty}D_k\sin\alpha_{k_2}x=Bx+\sum_{k=1}^{\infty}A_k\sin\alpha_{k_1}x \tag{1.8}$$

$$\sum_{k=1}^{\infty}C_k\sin\alpha_{k_1}x=-Bx+\sum_{k=1}^{\infty}B_k\sin\alpha_{k_2}x$$

The coefficients $D_k$ and $C_k$ are found from (1.8), respectively, in terms of $A_k$ and $B_k$

$$D_m = \frac{2}{l}\left[B\delta_{m_2} - \frac{(h_2 - h_1)}{h_1 h_2}\sum_{k=1}^{\infty}\delta_{k_1, m_2} A_k\right] \tag{1.9}$$

$$C_m = \frac{2}{l}\left[-B\delta_{m_1} + \frac{(h_2 - h_1)}{h_1 h_2}\sum_{k=1}^{\infty}\delta_{k_2, m_1} B_k\right] \qquad (m = 1, 2, 3, \ldots)$$

where the following notation is used:

$$\delta_{m_i} = s_i(1 + h_i l)\,\alpha_{m_i}^{-2}\sin\alpha_{m_i} l \tag{1.10}$$

$$\delta_{k_i, m_j} = s_j(\alpha_{k_i}^2 - \alpha_{m_j}^2)^{-1}\alpha_{k_i}\alpha_{m_j}\cos\alpha_{k_i} l\cos\alpha_{m_j} l$$

$$s_h = \left(1 - \frac{\sin 2\alpha_{m_h} l}{2\alpha_{m_h} l}\right)^{-1}$$

Obviously $\lim\delta_{k_i, m_j} = 0$ $(k_i = m_j \to \infty)$. After some transformations (1.9) reduces to an infinite system of linear equations defining the coefficients $C_m$ and $D_m$ $(m = 1, 2, \ldots)$

$$C_m = \mu_m + \sum_{j=1}^{\infty}\mu_{j, m} C_j, \qquad D_m = \vartheta_m + \sum_{j=1}^{\infty}\vartheta_{j, m} D_j \tag{1.11}$$

where

$$\mu_m = -\frac{2B}{l}\delta_{m_1} + \frac{4B}{l^2}\left(\frac{h_2 - h_1}{h_1 h_2}\right)\sum_{k=1}^{\infty}\delta_{k_2, m_1}\gamma_{k_2}\delta_{k_2}$$

$$\vartheta_m = \frac{2B}{l}\delta_{m_2} + \frac{4B}{l^2}\left(\frac{h_2 - h_1}{h_1 h_2}\right)\sum_{k=1}^{\infty}\delta_{k_2, m_1}\beta_{k_1}\delta_{k_1}$$

$$\mu_{j, m} = -\frac{4}{l^2}\left(\frac{h_2 - h_1}{h_1 h_2}\right)^2\sum_{k=1}^{\infty}\delta_{k_2, m_1}\gamma_{k_2}\delta_{j_1, k_2}\beta_{j_1} \tag{1.12}$$

$$\vartheta_{j, m} = -\frac{4}{l^2}\left(\frac{h_2 - h_1}{h_1 h_2}\right)^2\sum_{k=1}^{\infty}\delta_{k_1, m_2}\beta_{k_1}\delta_{j_2, k_1}\gamma_{j_2}$$

It is evident from (1.10) that the series appearing in the expressions for $\mu_m$ and $\vartheta_m$ are convergent, and $\mu_m \to 0$ and $\vartheta_m \to 0$ when $m \to \infty$. From (1.4) follows the convergence of series $\mu_{j,m}$ and $\vartheta_{j,m}$, also

$$\sum_{j=1}^{\infty}|\mu_{j, m}| \text{ and } \sum_{j=1}^{\infty}|\vartheta_{j, m}|$$

Hence, if $l^{-2}(h_2 - h_1)^2 h_1^{-2} h_2^{-2}$ is sufficiently small for the inequality

$$\sum_{j=1}^{\infty}|\mu_{j, m}| \leqslant 1 - \theta, \qquad \sum_{j=1}^{\infty}|\vartheta_{j, m}| \leqslant 1 - \theta \qquad (0 < \theta < 1)$$

to be satisfied, the infinite system of linear equations (1.11) is completely regular, and it is possible to determine coefficients $C_m$ and $D_m$ by the method of successive approximations [7]. Constants $T_1$ and $T$ are found from Eqs. (1.6). Similar results were obtained for an infinite region in [8] by a somewhat different procedure.

2. Let $h_1 = h_2 = h$. Equation (1.4) now becomes

$$\operatorname{tg}\alpha_k l = -\frac{1}{hl}\alpha_k l \tag{2.1}$$

Formulas (1.7) and (1.8) on the assumption that $h_1 = h_2 = h$ yield for coefficients

$C_k$ and $D_k$ the following expressions:

$$C_k = e_k \frac{\gamma_k - 1}{1 - \beta_k \gamma_k}, \quad D_k = e_k \frac{1 - \beta_k}{1 - \beta_k \gamma_k}, \quad e_k = \frac{2}{l}(q_1 - q_2) \frac{\sin \alpha_k l}{\alpha_k^2 \left(1 - \dfrac{\sin 2\alpha_k l}{2\alpha_k l}\right)} \tag{2.2}$$

$$\beta_k = \exp[-\alpha_k^2 a^2 T_1], \qquad \gamma_k = \exp[-\alpha_k^2 a^2 (T - T_1)]$$

If $h = 0$, then

$$\alpha_k l = {}^1/_2\pi(2k - 1), \qquad \sin 2\alpha_k l = 0 \tag{2.3}$$

The conditions of existence of roots $T_1$ and $T$ of these equations ($C_k \sin \alpha_k l > 0$ and $D_k \sin \alpha_k l < 0$)

$$-\frac{q_2 l}{1 + hl} < u_{**} < u_* < -\frac{q_1 l}{1 + hl} \tag{2.4}$$

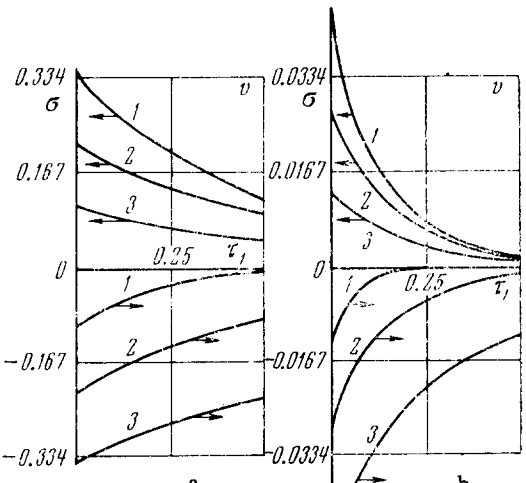
follow from formulas (2.2) and (1.3).

The patterns of functions $\sigma = \sigma(\tau_1)$ and $\vartheta = \vartheta(\tau_1)$ for the cases of $h = 0$ and $hl = 4$ are shown in Fig. 1 (a) and (b) respectively, on the assumption that along each of these curves $T = bT_1$. Curves 1, 2 and 3 relate to $b = 4$, 2 and $^4/_3$ respectively. Here

$$\sigma = \frac{\sigma'}{2(q_2 - q_1)l}, \quad \vartheta = \frac{\vartheta'}{2(q_2 - q_1)l}, \quad \tau_1 = \frac{a^2}{l^2} T_1, \quad \sigma' = -u_* - \frac{q_1 l}{1 + hl}$$

$$\vartheta' = -u_{**} - \frac{q_2 l}{1 + hl}$$

It is seen from formulas (2.2) and (1.6) that with increasing $T_1$ the values $\sigma$ decrease, while $\vartheta$ increase. Clearly, unique values of $T_1$ correspond to any $u_{**}$ and $u_*$ which satisfy inequalities (2.4).




a b

Fig. 1.

A solution of the problem of periodic modes of one-dimensional distributed system of temperature control of a furnace was given in [1] together with the analysis of stability of such modes. This problem reduces to solving Eq. (1.1) with conditions (1.2), where $x$ ($-l < x < 0$) is substituted for $-l$ and $h_1 = h_2 = 0$ is assumed.

**3.** Let us now examine the asymptotic behavior of the solution of Eq. (1.1) with conditions (1.2), in which $h_1 = h_2 = h$ is assumed, for $t \to \infty$ and initial condition

$$u(x, 0) = \varphi(x) \tag{3.1}$$

where $\varphi(x)$ is a function satisfying the Dirichlet condition in the interval $-l \leqslant x \leqslant 0$.

The solution of this problem is defined by formulas (similar to those derived in [6])

$$u_1^{(i+1)}(x, t) = \frac{q_1 x}{1 + hl} + \sum_{k=1}^{\infty} C_k^{(i+1)} \exp\left\{-\lambda_k^2 \left(t - \sum_{j=0}^{i} T^{(j)}\right)\right\} \sin \alpha_k x$$

$$\sum_{j=0}^{i} T^{(j)} \leqslant t \leqslant \sum_{j=0}^{i} T^{(j)} + T_1^{(i+1)} \qquad (i = 0, 1, 2, 3, \ldots), \qquad T^{(0)} = 0 \tag{3.2}$$

$$u_2^{(i+1)}(x, t) = \frac{q_2 x}{1 + hl} + \sum_{k=1}^{\infty} D_k^{(i+1)} \exp\left\{-\lambda_k^2\left(t - \sum_{j=0}^{i} T^{(j)} - T_1^{(i+1)}\right)\right\} \sin \alpha_k x$$

$$\sum_{j=0}^{i} T^{(j)} + T_1^{(i+1)} \leqslant t \leqslant \sum_{j=0}^{i} T^{(j)} + T^{(i+1)} \qquad (i = 0, 1, 2, 3, \ldots)$$

where

$$C_k^{(1)} = \left(\varphi_k - \frac{2q_1}{l} \frac{\sin \alpha_k l}{\alpha_k^2}\right)\left(1 - \frac{\sin 2\alpha_k l}{2\alpha_k l}\right)^{-1}$$

$$\varphi_k = \frac{2}{l} \int_{-l}^{0} \varphi(x) \sin \alpha_k x dx, \qquad D_k^{(i)} = e_k + \beta_k^{(i)} C_k^{(i)}$$

$$C_k^{(i+1)} = -e_k + \gamma_k^{(i)} D_k^{(i)} \qquad (i = 1, 2, 3 \ldots) \tag{3.3}$$

$$\beta_k^{(i)} = \exp[-\lambda_k^2 T_1^{(i)}], \qquad \gamma_k^{(i)} = \exp[-\lambda_k^2 (T^{(i)} - T_1^{(i)})]$$

For the sake of definiteness we assume that at the initial instant the upper inequality of the first of conditions (1.4) applies.

If $h = 0$, then $\beta_k^{(i)} = (\beta_1^{(i)})^{(2k-1)^2}$ and $\gamma_k^{(i)} = (\gamma^{(i)})^{(2k-1)^2}$, and it can be shown, as was done in [6] in the analysis of another problem, that with condition

$$\varphi(-l) < u_*, \ \sigma > 0, \ \vartheta < 0$$

satisfied, $\beta_1^{(i+1)}$ and $\gamma_1^{(i+1)}$ may in effect be determined from equations

$$\beta_1^{(i+1)} = \beta_1^{(0)} + A\{\beta_1^{(i+1)}(\gamma_1^{(i)})^9 + (\beta_1^{(i+1)})^9\}, \quad \gamma_1^{(i+1)} = \gamma_1^{(0)} + \\ + B\{\gamma_1^{(i+1)}(\beta_1^{(i+1)})^9 + (\gamma_1^{(i+1)})^9\} \tag{3.4}$$

$$\beta_1^{(0)} = \frac{\sigma}{\vartheta - e_1}, \quad \gamma_1^{(0)} = \frac{\vartheta}{\sigma + e_1}, \quad A = \frac{1}{9}\frac{e_1}{\vartheta - e_1}, \quad B = -\frac{1}{9}\frac{e_1}{\sigma + e_1}$$

(provided that $\beta_1^{(0)}$ and $\gamma_1^{(0)}$ are sufficiently small for considerable $i$)

Conditions $0 < \beta_1^{(0)} < 1$ and $0 < \gamma_1^{(0)} < 1$ yield

$$\vartheta > e_1, \ \sigma < -e_1, \ \sigma - \vartheta < -e_1, \ A < 0, \ B < 0$$

It follows from (3.4) that $\beta_1^{(i)}$ and $\gamma_1^{(i)}$ are bounded monotonic sequences which, according to the theory of Weierstrass have $\beta_1$ and $\gamma_1$ as their limits. Substituting these limits into formulas (3.3), we find that these reduce to (2.2). Thus solution (3.2) virtually becomes the solution of the periodic problem (1.3) in which use is made of (2.3).

Let now $h \neq 0$. In this case $\beta_k^{(i)} = (\beta_1^{(i)})^{\mu_k}$ and $\gamma_k^{(i)} = (\gamma_1^{(i)})^{\mu_k}$, where $\mu_k = (\alpha_k / \alpha_1)^2$. More general equations may be substituted for formulas (3.4). Let us consider the case in which these are of the form

$$\beta_1^{(i+1)} = \beta_1^{(0)} + \sum_{j=1}^{s} a_j^{(i+1)} [\beta_1^{(i+1)}]^{\mu_j}, \quad \gamma_1^{(i+1)} = \gamma_1^{(0)} + \sum_{j=1}^{s} b_j^{(i+1)} [\gamma_1^{(i+1)}]^{\mu_j}$$

$$\beta_1^{(0)} = \frac{\sigma}{\vartheta - e_1} \qquad \gamma_1^{(0)} = \frac{\vartheta}{\sigma + e_1} \tag{3.5}$$

$$a_1^{(i+1)} = \omega_1 (\gamma_1^{(i)})^{\mu_2} [1 - (\gamma_1^{(i)})^{\mu_2}] + \omega_2 (\gamma_1^{(i)})^{\mu_3}$$

$$a_2^{(i+1)} = \omega_2 [1 - (\gamma_1^{(i)})^{\mu_2} + (\gamma_1^{(i)}\beta_1^{(i)})^{\mu_2}], \qquad a_3^{(i+1)} = \omega_3 [1 - (\gamma_1^{(i)})^{\mu_3}]$$

$$\omega_2 = -\frac{\nu_2}{\nu_1 + \vartheta}, \qquad \omega_3 = -\frac{\nu_3}{\nu_1 + \vartheta}, \qquad \nu_k = -e_k \sin \alpha_k l, \quad s = 3$$

Formulas for $b_j^{(i+1)}$ are derived from corresponding formulas for $a_j^{(i+1)}$ by substituting $(\sigma - \nu_1)^{-1}$ for $(\nu_1 + \vartheta)^{-1}$, $\beta_1^{(i+1)}$ for $\gamma_1^{(i)}$ and $\gamma_1^{(i)}$ for $\beta_1^{(i)}$.

Formulas (3.5) imply the following inequalities:

$$a_j^{(i+1)} < 0, \ b_j^{(i+1)} < 0 \qquad (j = 1, 2, 3)$$

$$\beta_{\min} < \beta_1^{(i+1)} < \beta_{\max} = \beta_1^{(0)}, \qquad \gamma_{\min} < \gamma_1^{(i+1)} < \gamma_{\max} = \gamma_1^{(0)}$$

We must obviously have $0 < \gamma_1^{(0)} < 1$ and $0 < \beta_1^{(0)} < 1$ hence conditions that $0 < \sigma < \nu_1$, $-\nu_1 < \vartheta < 0$ and $\sigma - \vartheta < \nu_1$.

With the use of (3.5) we construct the difference

$$\beta_1^{(i+1)} - \beta_1^{(i)} = a_1^{(i+1)} [\beta_1^{(i+1)} - \beta_1^{(i)}] + \beta_1^{(i)} [a_1^{(i+1)} - a_1^{(i)}] + a_2^{(i+1)} (\beta_1^{(i+1)})^{\mu_2} - a_2^{(i)} (\beta_1^{(i)})^{\mu_2} + a_3^{(i+1)} (\beta_1^{(i+1)})^{\mu_3} - a_3^{(i)} (\beta_1^{(i)})^{\mu_3} \qquad (3.6)$$

Using the mean value theorem we define the difference $a_1^{(i+1)} - a_1^{(i)}$ as

$$a_1^{(i+1)} - a_1^{(i)} = \{\omega_2\mu_2\xi^{\mu_2-1} [1 - (\beta_1^{(i)})^{\mu_2}] + \omega_3\mu_3\delta^{\mu_3-1}\} (\gamma_1^{(i)} - \gamma_1^{(i-1)}) - \omega_2\mu_2\eta^{\mu_2-1} (\gamma_1^{(i-1)})^{\mu_2} (\beta_1^{(i)} - \beta_1^{(i-1)}) \qquad (3.7)$$

where $\xi$ and $\delta$ are certain mean values of $\gamma_1$ and $\eta$ is a certain mean value of $\beta_1$.

Similar formulas can be written for the remaining differences appearing in the expression (3.6), as well as for the terms of formula for the difference $\gamma_1^{(i+1)} - \gamma_1^{(i)}$ similar to (3.6), and which can be derived from (3.5). These formulas make it possible to obtain linear expressions for the differences $\beta_1^{(i+1)} - \beta_1^{(i)}$ and $\gamma_1^{(i+1)} - \gamma_1^{(i)}$ in terms of $\gamma_1^{(i)} - \gamma_1^{(i-1)}$, $\beta_1^{(i)} - \beta_1^{(i-1)}$ and $\beta_1^{(i+1)} - \beta_1^{(i)}$, $\gamma_1^{(i)} - \gamma_1^{(i-1)}$, respectively,

$$\beta_1^{(i+1)} - \beta_1^{(i)} = K (\gamma_1^{(i)} - \gamma_1^{(i-1)}) + L (\beta_1^{(i)} - \beta_1^{(i-1)}) \qquad (3.8)$$

$$\gamma_1^{(i+1)} - \gamma_1^{(i)} = M (\beta_1^{(i+1)} - \beta_1^{(i)}) + N (\gamma_1^{(i)} - \gamma_1^{(i-1)})$$

$$K = \frac{1}{E} (\omega_2\mu_2 \{\xi^{\mu_2-1}\beta_1^{(i)} - \zeta^{\mu_2-1} (\beta_1^{(i)})^{\mu_2}\} [1 - (\beta_1^{(i)})^{\mu_2}] + \omega_3\mu_3 \{\delta^{\mu_3-1}\beta_1^{(i)} - \Delta^{\mu_3-1} (\beta_1^{(i)})^{\mu_3}\})$$

$$L = \frac{\omega_2\mu_2}{E} \{-\eta^{\mu_2-1} \beta_1^{(i)} + \nu^{\mu_2-1} (\beta_1^{(i)})^{\mu_2}\} (\gamma_1^{(i-1)})^{\mu_2}$$

$$E = 1 - a_1^{(i+1)} - \omega_2\mu_2\psi^{\mu_2-1} \{1 - (\gamma_1^{(i)})^{\mu_2} [1 - (\beta_1^{(i)})^{\mu_2}]\} - \omega_3\mu_3\vartheta^{\mu_3-1} [1 - (\gamma_1^{(i)})^{\mu_3}]$$

where $\xi$, $\zeta$, $\Delta$ and $\delta$ are mean values of $\gamma_1$; $\eta$, $\nu$, $\psi$ and $\vartheta$ are the mean values of $\beta_1$. Expressions for $M$ and $N$ are derived from the formulas for $K$ and $L$ with the use of above substitutions.

Estimates

$$|K| < \alpha, \quad |L| < \beta, \quad |M| < \gamma, \quad |N| < \delta$$

yield inequalities applicable to system (3.8) for $i \geqslant k$, where $k$ is a certain constant

$$|\beta_1^{(i+1)} - \beta_1^{(i)}| < \alpha |\gamma_1^{(i)} - \gamma_1^{(i-1)}| + \beta |\beta_1^{(i)} - \beta_1^{(i-1)}|$$

$$|\gamma_1^{(i+1)} - \gamma_1^{(i)}| < \gamma |\beta_1^{(i+1)} - \beta_1^{(i)}| + \delta |\gamma_1^{(i)} - \gamma_1^{(i-1)}| \qquad (3.9)$$

where

$$\alpha = \frac{1}{F}\left[-\omega_2\mu_2\gamma_{\max}^{\mu_2-1}\beta_{\max}\left(1-\beta_{\max}^{\mu_2-1}\right)\left(1-\beta_{\min}^{\mu_2}\right)-\omega_3\mu_3\gamma_{\max}^{\mu_3-1}\beta_{\max}\left(1+\beta_{\max}^{\mu_3-1}\right)\right]$$

$$\beta = \frac{-\omega_2\mu_2}{F}\left(\gamma_{\max}\beta_{\max}\right)^{\mu_2}\left(1+\beta_{\max}^{\mu_2-1}\right) \qquad (3.10)$$

$$F = 1-\omega_2\gamma_{\min}^{\mu_2}\left(1-\beta_{\max}^{\mu_2}\right)-\omega_3\gamma_{\min}^{\mu_3}-$$

$$-\omega_2\mu_2\beta_{\min}^{\mu_2-1}\left\{1-\gamma_{\max}^{\mu_2}\left(1-\beta_{\min}^{\mu_2}\right)\right\}-\omega_3\mu_3\beta_{\min}^{\mu_3-1}\left(1-\gamma_{\max}^{\mu_3}\right)$$

and $\gamma$ and $\delta$ are defined by similar expressions. In this case the inequality $F>0$, where $F$ is defined by formula (3.10), must be satisfied.

Multiplying the first of inequalities (3.9) by $\xi$ and the second by $\eta$, where $\xi$ and $\eta$ are certain unknown, a priori positive numbers, and adding the two products, we obtain

$$\eta\left|\gamma_1^{(i+1)}-\gamma_1^{(i)}\right|+(\xi-\gamma\eta)\left|\beta_1^{(i+1)}-\beta_1^{(i)}\right|<$$
$$<(\alpha\xi+\delta\eta)\left|\gamma_1^{(i)}-\gamma_1^{(i-1)}\right|+\beta\xi\left|\beta_1^{(i)}-\beta_1^{(i-1)}\right| \qquad (3.11)$$

We denote $\zeta=\xi-\gamma\eta$ and seek the numbers $\xi$ and $\eta$ as the solution of the system of homogeneous linear equations

$$\alpha\xi+\delta\eta=\lambda\eta, \qquad \beta\xi=\lambda\zeta=\lambda(\xi-\lambda\eta) \qquad (3.12)$$

Inequality (3.11) now becomes

$$\eta\left|\gamma_1^{(i+1)}-\gamma_1^{(i)}\right|+\zeta\left|\beta_1^{(i+1)}-\beta_1^{(i)}\right|<\lambda\left\{\eta\left|\gamma_1^{(i)}-\gamma_1^{(i-1)}\right|+\right.$$
$$\left.+\zeta\left|\beta_1^{(i)}-\beta_1^{(i-1)}\right|\right\} \qquad (3.13)$$

If the roots $\xi$ and $\eta$ of this system of equations could be selected so that

$$\xi>0, \eta>0, \zeta>0, 0<\lambda<1$$

then the numbers $\zeta\beta_1^{(i)}$ and $\eta\gamma_1^{(i)}$ could be considered as elements $x^{(i)}=\{\zeta\beta_1^{(i)}, \eta\gamma_1^{(i)}\}$ of space $l_1^{(2)}$ [9] of range $\rho(x^{i+1},x^i)=|\zeta(\beta_1^{(i+1)}-\beta_1^{(i)})+|\eta(\gamma_1^{(i+1)}-\gamma_1^{(i)})$. Let us assume this to be possible. From (3.13) then follows that $\rho(x^{i+1},x^i)<\lambda\rho(x^i,x^{i-1})$. We further obtain inequality

$$\rho(x^m, x^{m+q})<\frac{\lambda^{m-k}}{1-\lambda}\rho(x^k, x^{k-1})$$

i.e., $\qquad \rho(x^m, x^{m+q})\to 0 \quad \text{for } m\to\infty,\ q>0.$

Hence sequence $\{x^i\}$ is convergent in itself, and by virtue of completeness of space $l_1^{(2)}$ there exists an element $x^c=\{\zeta\beta_1, \eta\gamma_1\}\in l_1^{(2)}$ which is the limit of this sequence. After substitution of $\beta_k^{(i)}$ and $\gamma_k^{(i)}$ for $\beta_k$ and $\gamma_k$ respectively, into formulas (3.2) and (3.3) these again virtually become (2.2) and (1.3).

Let us prove that it is possible to find such $\xi$, $\eta$, $\zeta$ and $\lambda$ which satisfy the inequalities $\xi>0$, $\eta>0$, $\zeta>0$ and $0<\lambda<1$.

The system of Eqs. (3.12) is linear and homogeneous. For the existence of a nontrivial solution of this system its determinant must be zero, and this yields for the determination of $\lambda$ a quadratic equation whose roots are

$$\lambda={}^1/_2\left\{\beta+\delta+\alpha\gamma\pm\sqrt{(\beta+\delta+\alpha\gamma)^2-4\beta\delta}\right\} \quad (\lambda_1<\lambda_2)$$

It is readily seen that both roots are real and positive numbers (since $\alpha$, $\beta$, $\gamma$ and $\sigma$ are positive). If condition

$$\beta+\delta+\alpha\gamma-\beta\delta<1 \qquad (3.14)$$

is satisfied, both roots are smaller than unity.

We note that, when considering Eqs. (3.4) for the case of $h = 0$, the relevant condition is of the form $\alpha\gamma < 1$. Assuming $\eta > 0$, we further obtain from Eqs. (3.12) that $\zeta > 0$ and $\xi = (\lambda - \delta)\eta\alpha^{-1} > 0$. This means that $\lambda_1 < \delta$ and $\lambda_2 > \delta$. Bearing this in mind, we set $\lambda = \lambda_2$ and $\xi = (\lambda_2 - \delta)\eta\alpha^{-1}$. Condition (3.13) in which $0 < \lambda < 1$ is now satisfied.

It can be proved by the method of consecutive approximations that for sufficiently small $\beta_1^{(0)}$ and $\gamma_1^{(0)}$ the increments resulting from the substitution of shortened equations for Eqs. (3.5) $(s \to \infty)$ also tend to limits. The limits to which solutions of exact equations tend are unique.

Thus at the limit at $t \to \infty$ with conditions $\varphi(-l) < u_*$, $0 < \sigma < \nu_1$, $-\nu_1 < \vartheta < 0$, $\sigma - \vartheta < \nu_1$ and $F > 0$, where $F$ is defined by formula (3.10), and the inequality (3.14) satisfied the solution $u(x, t)$ of the considered problem tends to the periodic solution of the related problem.

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